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THE CONTACT STRESS PROBLEM FOR AN ELASTIC SPHERE INDENTING AN ELASTIC CAVITY

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Abstract—An approximate solution is given for the contact stress problem of an elastic sphere indenting an elastic cavity. For this problem the usual assumption of a small contact region is not made and the subsequent results are compared with the solution as given by H. Hertz. Improvements to the Hertz theory that are within the framework of classical elasticity theory are discussed.

1. INTRODUCTION

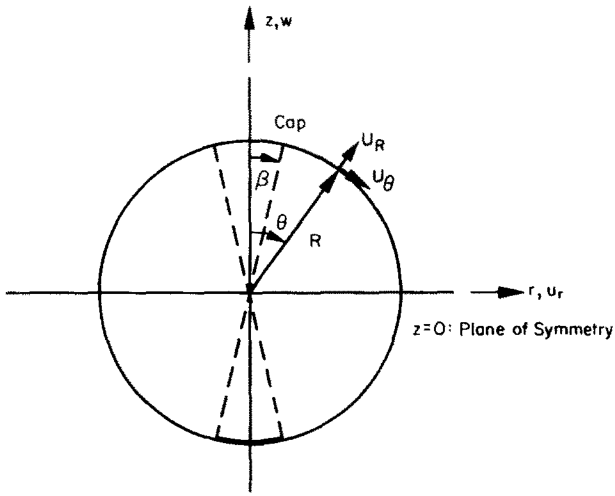
THE Hertz [1] analysis of stresses and deformations in the contact region of two ellipsoidal bodies pressed against one another assumes that the contact region is so small compared with the smallest principal radius of curvature of the undeformed ellipsoids at the initial point of contact that the stresses and deformations may be determined by methods appropriate to the problem of the plane in the linear theory of elasticity. In the present discussion we remove this restriction, while retaining the assumptions that stresses and deformations are small enough to permit the analysis to be conducted within the framework of the linear theory of elasticity. On so doing it becomes possible to analyze such questions as the state of stress in a sphere pressed into a cavity of slightly larger radius.

As applied to a sphere pressed into a spherical cavity, the Hertz analysis considers two spherical surfaces of arbitrary radii and arbitrary elastic constants compressed by forces acting along the line connecting their centers and normal to the tangent plane at the initial point of contact. In the analysis considered herein it is assumed, for convenience of computation, that the elastic properties of the sphere and of the cavity are identical. The intention here is to solve this problem in a manner appropriate to the sphere.

In order to avoid misunderstanding it should be said that the assumption of a small contact region is used in the Hertz theory (a) to justify calculation of deformations by methods appropriate to the problem of the plane and, (b), to justify replacing the spherical surfaces by their osculating paraboloids of revolution. The latter use is merely a mathematical convenience. In the sequel a discussion describes why higher approximations to the spherical surface that remove the use (b) are mathematically inconsistent with retention of assumption (a) and why for any extension of the Hertz theory to be significant the difference between the radii of sphere and cavity must be small.

2. METHOD OF SOLUTION

Certain symmetry properties of loading are utilized for the solution to this problem. The stresses on an individual sphere are axially symmetric about the line containing the center of the sphere and of the cavity. Further symmetry is ensured by considering the stresses on a given sphere to be symmetric about a meridional plane that contains the center of a sphere and is perpendicular to a line of axial symmetry.



β=Angle of contact between spheres

FIG. 1. Geometry and coordinate system for a sphere.

As in the Hertz theory the shear stress at the interface under consideration will be assumed to be zero, i.e. the spheres and cavity are smooth. The boundary conditions are derived as follows. The surface of the sphere that lies within twice the angle of contact, β, which is defined in Fig. 1, is denoted the 'cap' of the sphere and is the portion of the sphere that comes in contact with the adjacent surface of the cavity. The boundary conditions corresponding to the portion of the sphere exterior to the caps are as follows:

$$\left.\begin{aligned} \sigma_{RR} &= 0 \\ \sigma_{R\varphi} = \sigma_{R\theta} &= 0 \end{aligned}\right\} \beta < \theta < \pi - \beta, \tag{1}$$

where it is natural to express the stresses in spherical coordinates (R, θ, φ). For that portion of the surface within the cap region region, the stress boundary condition is

$$\sigma_{R\theta} = \sigma_{R\varphi} = 0 \begin{cases} 0 \leq \theta \leq \beta, \\ \pi - \beta \leq \theta \leq \pi. \end{cases} \tag{2}$$

For the region within the caps it is convenient to express the displacements in cylindrical coordinates (r, θ, z). If the displacements on the sphere and on the cavity are labeled w_1 and w_2 respectively in cylindrical coordinates, then the actual movement of points on the surface relative to points on the perimeter is given by

$$\begin{aligned} &w_1 - w_2^0, \\ &w_2 - w_2^0, \end{aligned} \tag{3}$$

where w_1^0 represents the displacement of the ring of points on the perimeter of the contact region. The displacement boundary condition for each sphere in contact with the surface of the cavity may be written

$$\begin{aligned} |w_1 - w_1^0| + |w_2 - w_2^0| \frac{\sin\beta_1}{\sin\beta_2} &= -(\cos\theta_2 - \cos\beta_2)\frac{\sin\beta_1}{\sin\beta_2} + (\cos\theta_1 - \cos\beta_1), \\ \frac{\sin\beta_1}{\sin\beta_2} &= \frac{R_2}{R_1}, \end{aligned} \tag{4}$$

where all displacements are considered positive and w_i are dimensionless. It now remains to relate the known displacements in the cap region to the unknown stresses there. This is done by assuming the stresses to be known within the cap and solving for the displacements by means of one of the well-known solutions to the problem of the sphere. Putting the displacements obtained from the unknown stresses into the displacement boundary conditions, an integral equation is established. The solution to this integral equation determines the stresses within the cap and hence over the entire surface of the sphere, thereby solving the problem.

The solution to the first boundary value problem used in this work is based on the Boussinesq solutions to the field equations of elasticity for zero body forces and temperature changes. The particular type of solution used is that developed by Sternberg *et al.* [2] for the solution of the problem of the hollow sphere with axisymmetric loading. By taking the Lamé elastic constants, λ and μ, to be equal, a further simplification is made. This does not seriously restrict the results (it is equivalent to taking Poisson's ratio to be 0·25) and allows for considerable reduction in the calculations. The displacements in the z-direction from this solution are given as

$$\begin{aligned} w_i &= \sum A_n \left\{ \alpha_{in} \left[\frac{n+1}{2n+1} P_{n+1} + \frac{n}{2n+1} P_n \right] + \gamma_{in} \left[\frac{n(n+1)}{2n+1} (P_{n+1} - P_{n-1}) \right] \right\}, \\ P_v &= P_v(\cos\theta) = P_v(t), \end{aligned} \tag{5}$$

where

$$A_n = (2n+1) \int_{-1}^{1} \sigma_{RR}(t) P_n(t)\, \mathrm{d}t$$

and

$$\begin{aligned} \alpha_{1n} &= (3n^2 + n/2 - 1)/[(n-1)(n^2 + \tfrac{3}{2}n + 5/4)], \\ \gamma_{1n} &= (n + 7/2)/[(n-1)(n^2 + \tfrac{3}{2}n + 5/4)], \end{aligned}$$

for a sphere and

$$\begin{aligned} \alpha_{2n} &= (3n^2 + \tfrac{11}{2}n + 3/2)/[2(n+2)(n^2 + n/2 + 3/4)], \\ \gamma_{2n} &= -(n + 9/2)/[2(n+2)(n^2 + n/2 + 3/4)], \end{aligned}$$

for a spherical cavity in an infinite space. Introducing the operator, $A_i(\theta, \beta)$, by the defining expression

$$A_i(\theta,\beta) = \sum A_n\left\{\alpha_{in}\left[\frac{n+1}{2n+1}P_{n+1}+\frac{n}{2n+1}P_n\right]+\gamma_{in}\left[\frac{n(n+1)}{2n+1}(P_{n+1}-P_{n-1})\right]\right\}^{(i=1,2)} \quad (6)$$

the following more compact form for equations (3) is developed:

$$w_i - w_i^0 = A_i(\theta,\beta) - A_i(\beta,\beta), \quad (7)$$

$A_i(\theta, \beta)$ depends upon the angle of contact, β, since the integral contained in $A_i(\theta, \beta)$ is non-zero for $0 \leq \theta \leq \beta$, $\pi - \beta \leq \theta \leq \pi$. The operator, $A_i(\theta, \beta)$, is also a function of the running variable, θ. The appropriate quantities, $w_i - w_i^0$, placed into equation (4), form

$$f(\theta) = \int_0^\beta K(\theta, t)\sigma_{RR}(t)\,\mathrm{d}t \quad (8)$$

where $K(\theta, t)$ is a kernel made up of the quantities described by equations (5).

The problem is therefore completely formulated within the assumptions mentioned previously. In the cap the values of displacements are prescribed and it is required to find a distribution of stress that will satisfy the boundary conditions, (4). The integral equation is a Fredholm integral equation of the first kind. The sequel will discuss the method used to obtain a solution asymptotic to the correct one for a sphere indenting a cavity of nearly the same radius.

To solve the integral equation, (7), a scheme was developed whereby a sequence of functions, converging rapidly to an exact solution, was used as an asymptotic representation of the correct stress distribution. Each term in the sequence had the following properties:

(a) Continuous stress over the entire surface of the sphere.
(b) Stresses decreasing from a maximum at $\theta = 0$ to zero at $\theta = \beta$.

The Hertz solution appropriate to the plane satisfies these conditions and it might be anticipated that a power series in which the Hertz solution was the leading term would converge very rapidly. However, using the Hertz solution for σ_{RR} in the integrals of equation (5) does not yield a result in terms of elementary functions. The function

$$\sigma_{RR} = (\cos\theta - \cos\beta)^{\frac{1}{2}} \quad (9)$$

is preferable as a leading term. The integral of the type used in equation (5) is given as [3], [4]·

$$\int_1^{\cos\beta} (t-\cos\beta)^{\frac{1}{2}} P_n(t)\,\mathrm{d}t = \frac{2^{\frac{1}{2}}}{2n+1}\left[\frac{\sin(n-1/2)\beta}{n-1/2} - \frac{\sin(n+3/2)\beta}{n+3/2}\right]. \quad (10)$$

From the standpoint of numerical computation this result is useful since all quantities of interest may be expressed in terms of elementary functions. One anticipates that a power series made up of terms given by

$$\sigma_{RR} = \sum_{m=1}^{\infty} B_m(\cos\theta - \cos\beta)^{m/2} \quad (11)$$

will satisfy the integral equation to any desired degree of accuracy. The computation of the integrals for $m = 1$ has been shown. For m odd the integral may be put into the form of equation (10) by integration by parts. For m even the integral can be obtained through the standard recursion formulae for the Legendre polynomials.

3. NUMERICAL SOLUTION AND RESULTS

The integral equation, (8), was solved in the following manner. Representation (11) was used for the normal stress distribution with $B_m = 0$ for $m > 3$. By defining $K_m(\theta, t)$ as that part of the kernel of equation (8) corresponding to the mth approximation of the normal stress, the integral equation becomes

$$f(\theta) = \sum_{m=1}^{\infty} B_m \int_0^{\beta} K_m(\theta, t)(\cos t - \cos \beta)^{m/2}\, \mathrm{d}t. \tag{12}$$

Equidistant data for θ was used in the computation of $K_m(\theta, t)$ with no less than ten divisions of θ for each β. The integral

$$\int_0^{\beta} K_m(\theta, t)(\cos t - \cos \beta)^{m/2}\, \mathrm{d}t \tag{13}$$

was computed so that

$$\max_{\theta \le \beta} \frac{\left|\int_0^{\beta} K_m(\theta, t)(\cos t - \cos \beta)^{\frac{1}{2}}\, \mathrm{d}t\right|}{\left|\int_0^{\beta} K_m(0, t)(\cos t - \cos \beta)^{\frac{1}{2}}\, \mathrm{d}t\right|} \le 0{\cdot}001 \tag{14}$$

for all m. The constants B_m were solved for by use of the method of least squares [5]. In all cases equation (12) was satisfied so that

$$\max_{\theta \le \beta} \frac{\left|f(\theta) - \int_0^{\beta} K(\theta, t)\sigma_{RR}(t)\, \mathrm{d}t\right|}{\left|\int_0^{\beta} K(0, t)\sigma_{RR}(t)\, \mathrm{d}t\right|} \le 0{\cdot}0005. \tag{15}$$

The results for the stress distribution for $R_1/R_2 = 1{\cdot}01$ and $\nu = 0{\cdot}25$ are presented in Table 1 below:

TABLE 1

Angle of contact	$B_{1/2\mu}$	$B_{2/2\mu}$	$B_{3/2\mu}$
5°	0·00613	−0·00059	0·04300
10°	0·00628	−0·00029	0·00590
15°	0·00627	−0·00022	−0·00023
20°	0·00646	−0·00021	−0·001750
25°	0·00682	−0·00025	−0·00201

Further results for these radii of curvature are shown in Figs. 2 and 3. Figure 2 shows the relative approach versus load. The relative approach, α, is the total displacement of the center of one of the spheres relative to the other center, $(|w_1| + |w_2|)_{\theta = 0^\circ}$. The

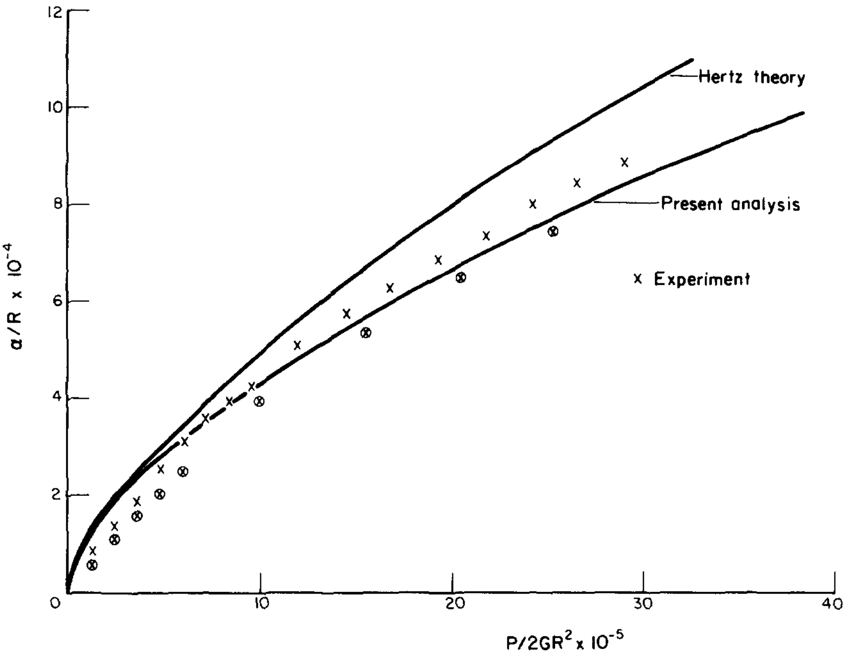


FIG. 2. Relative approach versus load for a sphere inside a spherical cavity (R_1/R_2 = 1·01).

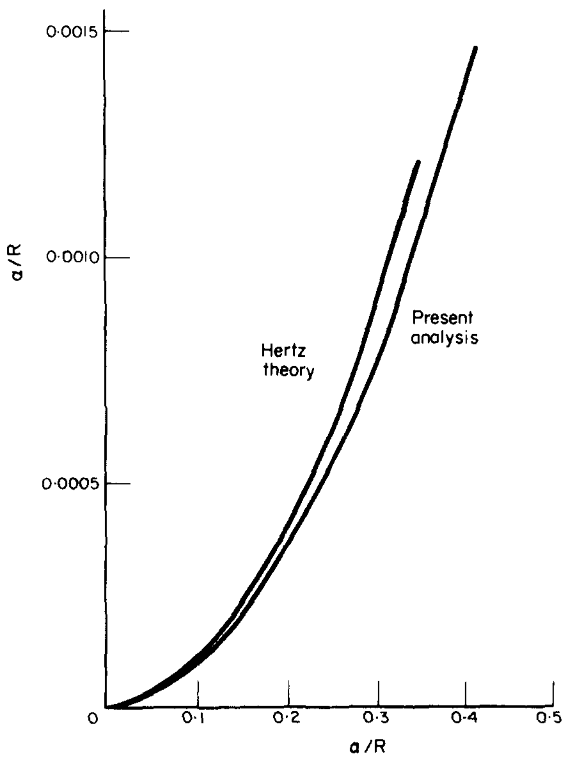


FIG. 3. Relative approach versus radius of contact region.

functional relationship between α and the load, P, may be written in the form:

$$\alpha/R = f(P/2\mu R^2). \tag{16}$$

The functional relationship, given in Fig. 2, compares the values obtained from the present theory with those computed from the Hertz theory. The results show a considerably harder spring relationship than is given by the Hertz theory. Figure 3 shows the relation between the angle of contact, β, and relative approach, α. These calculations were checked by measurements made on specimens of Duralumin HE 14T (Cu 3·5%–5·0%, ref. BS 1476/1949) for which Poisson's ratio was found to be 0·33 and the shear modulus was found to be $3{\cdot}7\times10^6$ psi. The concave element had a radius of 5·0393 in. and the convex one a radius of 4·9897 in. The measurements of relative approach versus load are shown in Fig. 2. For all but the lowest loads, where the measurement of relative approach is at minimum accuracy, the agreement with the present analysis is satisfactory.

4. IMPROVEMENTS TO HERTZ THEORY

Here our interest will be to consider extensions or improvements to the Hertz theory that are within the framework of classical elasticity theory. It will first be shown that if the spheres have greatly dissimilar radii, then the Hertz assumptions must be used. An extension to the theory in this case will require a finite strain analysis. Then the problem considered in the preceding Section will be shown to be a problem of classical elasticity appropriate to the sphere.

4.1. *Second order theory when ratio of radii differ greatly*

For this Section it is important to distinguish between a second order theory in which the assumptions of Hertz are retained and that of a second order theory which has as its basis the fundamental solutions of elasticity for the sphere. Consider the former solution and for the sake of argument specialize to the case where a solid sphere, radius R, indents a frictionless elastic half-space. The fundamental assumption is that the radius of the contact area is small compared with the radius of the sphere and therefore methods appropriate to the plane may be used. In this case equation (4) reduces to the following boundary condition

$$w-w^0 = R(\cos\theta-\cos\beta) \cong (1/2R)(a^2-r^2). \tag{17}$$

A solution for the plane that will solve this problem is given in [6]. From it we find that on the disk, $0 \le r \le a$, $z = 0$:

$$\begin{gathered}\int_0^r \frac{g(t)\,\mathrm{d}t}{(r^2-t^2)^{\frac{1}{2}}} = w(r),\\ \frac{\mu}{(1-\nu)}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{tg(t)\,\mathrm{d}t}{(t^2-r^2)^{\frac{1}{2}}} = \sigma_{zz}(r),\end{gathered} \tag{18}$$

where

$$g(t) = \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{rf(r)\,\mathrm{d}r}{(t^2-r^2)^{\frac{1}{2}}}.$$

Putting boundary condition, (17), into equation (18), one obtains the following result for the stress distribution within the disk, $0 \leq r \leq a$, $z = 0$:

$$\sigma_{zz}(r) = \frac{2\mu}{\pi(1-\nu)} \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \int_r^a \frac{[R-(t/2)\log|(R+t)/(R-t)|-(R^2-a^2)^{\frac{1}{2}}]t\,\mathrm{d}t}{(t^2-r^2)^{\frac{1}{2}}}$$
$$= \frac{4\mu}{\pi(1-\nu)}\left[\frac{(a^2-r^2)^{\frac{1}{2}}}{R} + \mathrm{O}\left(\frac{a^2}{R^2}\right)\right]. \tag{19}$$

Hence, it is easy to obtain a solution appropriate to the plane that is 'exact'. Difficulty, however, arises when the solution (19) is expanded in a Taylor series in the smallness parameter a/R, where $a/R \ll 1$. In this case a power series is obtained that contains terms which are of the same order of magnitude as those given by equation (11) for the sphere. By computing higher order terms appropriate to the plane, therefore, one neglects terms of the same order of magnitude that arise when considering the problem from the standpoint of the sphere.

Similarly, if one chooses to construct a solution appropriate to the sphere, difficulty is encountered unless the further assumption is made that the higher order elastic constants are identically zero. This is clear since the expansion of displacement boundary conditions in $a/R = \varepsilon$ will involve terms of a higher order of ε than the terms used in the Hertz theory. These higher order terms will involve elastic constants associated with second-order elasticity theory and therefore the problem is not one that is appropriate to classical elasticity theory. The authors wish to thank Professors A. E. Green and W. D. Collins for bringing this point to their attention.

4.2. *Justification for extension of Hertz theory in present problem*

This Section will show that one is justified in removing the restriction that the radius of the contact region is small in comparison with the smallest principal radius of curvature if the two radii of curvature are nearly the same in absolute magnitude but are of opposite sign. The displacement boundary conditions will be seen to depend upon a smallness parameter which will cause them to be similar to the Hertz boundary conditions but for a relatively large angle of contact.

The argument will consider the right-hand side of equation (4). The ratio of the principal radius of curvatures of sphere and cavity are given as

$$\frac{\sin\beta_1}{\sin\beta_2} = \frac{R_2}{R_1} = 1+\varepsilon \tag{20}$$

where $\varepsilon \ll 1$. The angle, θ_2, is put in terms of θ_1 and the right-hand side of equation (4) is expanded in a power series in ε. The following results are given for the cosine terms, retaining only terms of $\mathrm{O}(\varepsilon)$:

$$\frac{R_2}{R_1}\cos\theta_2 = \frac{(R_1^2-r^2)^{\frac{1}{2}}}{R_1} + \varepsilon\frac{R_1}{(R_1^2-r^2)^{\frac{1}{2}}} + \mathrm{O}(\varepsilon^2),$$
$$\frac{R_2}{R_1}\cos\beta_2 = \frac{(R_1^2-a^2)^{\frac{1}{2}}}{R_1} + \varepsilon\frac{R_1}{(R_1^2-a^2)^{\frac{1}{2}}} + \mathrm{O}(\varepsilon^2). \tag{21}$$

The right-hand side of equation (4) can now be rewritten as follows:

$$
\begin{aligned}
&(\cos\theta_2-\cos\beta_2)\frac{R_2}{R_1}-(\cos\theta_1-\cos\beta_1) = \\
&= -\varepsilon R_1\left[\frac{1}{(R_1^2-r^2)^{\frac{1}{2}}}-\frac{1}{(R_1^2-a^2)^{\frac{1}{2}}}\right]+0(\varepsilon^2).
\end{aligned} \tag{22}
$$

Equation (22) is seen to be of order ε, hence if the radius of contact is large compared with the depth of penetration of the sphere it may, in fact, be varied independently of this depth. If the radius of contact is small compared to the smallest principal radius of curvature so that $a/R = 0(\varepsilon)$, then equation (22) reduces to

$$
(\cos\theta_2-\cos\beta_2)\frac{R_2}{R_1}-(\cos\theta_1-\cos\beta_1) = \frac{\varepsilon}{2}\left(\frac{a^2-r^2}{R_1^2}\right)+0(\varepsilon^3) \tag{23}
$$

which is the boundary condition for the Hertz theory.

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Zusammenfassung—Das Problem der Berührungsbeanspruchung einer elastischen Kugel, welche einen elastischen Hohlraum eindrückt, wird annähernd gelöst, wobei bei diesem Problem die übliche Annahme einer kleinen Berührungszone nicht gemacht wird. Die Ergebnisse werden mit den von H. Hertz angegebenen Lösungen verglichen. Die innerhalb der klassischen Elastizitätstheorie fallenden Verbesserungen der Hertz'schen Theorie werden erörtert.

Абстракт—Дается примерное решение проблемы контактных напряжений для эластичной сферы, вдавливающей упругую плоскость. Для этой проблемы не делается обычного допущения о небольшом участке контакта, и последующие результаты сравниваются с решением данным Г. Герцем. Обсуждаются улучшения теории Герца, находящиеся в пределах классической теории упругости.